Qualifying Exam. Complex Analysis. Spring 2019

Problem 1. Does there exists a function f holomorphic in the unit disk and such that

$$f\left(2^{-2n}\right) = 2^{-2n} \qquad \text{and} \qquad f\left(2^{-(2n-1)}\right) = -2^{-(2n-1)}$$

for all positive integers n?

Problem 2. Let f be a holomorphic function that maps the unit disc into itself. Suppose that $w \in \mathbb{D}$ and $f(w) = 0$. Prove that

$$|f'(w)| \leq \frac{1}{1 - |w|^2}.$$

Problem 3. Does there exist an entire function f such that

$$\operatorname{Re} f(z) = (\operatorname{Im} z)^4, \quad z \in \mathbb{C}?$$

Problem 4. Evaluate the integral
$$\int_C \frac{\zeta^2}{(1-2\zeta)^2(2+3\mathrm{i}\zeta)}\,d\zeta,$$
where C is the unit circle oriented counter clockwise.

Problem 5. Evaluate the integral
$$\int_0^\infty \frac{\sin x}{x(1+x^2)} dx.$$